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On Ramond-Ramond fields and K-theory

Daniel S. Freed*

Department of Mathematics, University of Texas at Austin
Email: dafr@math.utexas.edu

Michael Hopkins

Department of Mathematics, Massachusetts Institute of Technology
Email: mjh@math.mit.edu

Abstract: A recent paper by Moore and Witten [1] explained that Ramond-Ramond fields in type-II superstring theory have a global meaning in K-theory. In this note we amplify and generalize some points raised in that paper. In particular, we express the coupling of the Ramond-Ramond fields to D-branes in a K-theoretic framework and show that the anomaly in this coupling exactly cancels the anomaly from the fermions on the brane, both in type IIA and type IIB.

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Contents

The proper quantization condition for the Ramond-Ramond (RR) field strengths in type-II superstring theory has been the subject of numerous studies, most recently a paper by Moore and Witten [1]. They assert that RR fields have a characteristic class in integral K-theory, in the same sense that a 1-form field which globally is a U(1) connection has a first Chern class in second integral cohomology. They also define the partition function of these fields, based on [2], and discuss the coupling to D-branes in its standard expression with differential forms and the resulting anomaly (in type IIA). Our main goal is to explain that this coupling is most naturally expressed in the K-theoretic framework (eq. (5.2) below) and that the coupling term has an anomaly which cancels the anomaly from fermions on the D-brane. This anomaly cancellation does not involve the RR partition function or the quadratic form needed to define it. It works in both type IIA and type IIB, though the details depend on the dimension of the D-brane. The cancellation holds for local and global anomalies.

We begin with a baby example from ordinary electromagnetism in four dimensions which we find useful in thinking about self-dual fields (better: fields with self-dual field strength). Then we review standard material about electric and magnetic coupling for p-form fields. For self-dual p-forms we make the simple observation that an electrically charged object is also magnetically charged and *vise versa*. Next, we hint at the correct mathematical framework which mixes integral K-theory and differential forms — a differential-geometric form of K-theory, which we call *differential* K-theory — so is the proper home for RR fields. This is the subject of a forthcoming joint paper with I. Singer. In the presence of D-branes, because of the magnetic charge there is a shift in the meaning of the RR fields; some of the details are dictated by the anomaly cancellation. Finally, we express the electric coupling of RR fields and D-branes in geometric K-theory and compute the anomaly.[1] The anomaly cancellation uses a geometric form of the Atiyah-Singer index theorem for families of real Dirac operators [3].

Our entire discussion assumes that the B-field of type II vanishes.

We are not careful with factors of 2π; the qualitative ideas discussed here do not depend on them and in any case they depend on conventions. On the other hand, certain factors of $1/2$ play a crucial role. A paper of Cheung and Yin [4] was instrumental in our understanding of these factors.

Cheung and Yin also demonstrate the perturbative anomaly cancellation for type-IIB D-branes.[2] Their work refines an earlier paper of Green, Harvey, and Moore [5]. A simultaneous paper of Moore and Minasian [6] also discusses the perturbative anomaly cancellation. They go further in that they also suggest the connection of D-brane charge to K-theory. The global anomaly was not treated in these references, and in [1] it was only discussed in a particular example.

Our lagrangians often include both a field and its dual, or self-dual fields. In lorentzian classical field theory, for example on Minkowski spacetime, self-duality may be imposed as an external constraint not derived from an action principle. The lagrangian gives Poisson brackets and one can proceed with canonical quantization in this framework. In euclidean quantum field theory, where one computes partition functions and correlation functions using the functional integral, the self-duality constraint is not imposed on the classical fields but rather one imposes it in defining the functional integral. In this paper we work in euclidean field theory, so do not impose the self-duality constraint on the classical fields; as we do not integrate over those fields we never encounter the subtleties of their quantization.

Anomaly cancellation is not sufficient to define correlation functions. Geometrically, the exponentiated (effective) action is naturally a section s of a hermitean line bundle with connection over a space S of fields, and correlation functions are integrals over S. The absence of anomalies is the assertion that the line bundle admits a flat section $\mathbb{1}$ of unit norm, and then one takes the exponentiated action to be the ratio $s/\mathbb{1}$. Now $\mathbb{1}$ is unique up to a phase on each connected component of S. The overall phase is irrelevant, but relative phases are important.[3] In our case the existence of an isomorphism between the line bundle for the fermion pfaffian and the (inverse) line

[1] We only sketch the argument here and leave a detailed proof for the subsequent paper.

[2] At the end of §4 in [4] there remains a puzzle about the D3-brane in type IIB. Our approach to the anomaly resolves it. In fact, since the Ramond-Ramond field in our formulation has self-dual field strength, *all* D-branes are both electrically and magnetically charged.

[3] They arise in many contexts, often as "θ-angles". As with other ingredients in field theory, they are constrained by locality.

bundle for the coupling of the RR fields and the D-brane is the index theorem. It appears that the index theorem can be strengthened to a *canonical* isomorphism. This would fix a trivializing section $\mathbb{1}$ and eliminate potential ambiguities in the definition of this part of the effective action. We hope to resolve this issue in a future paper.

1. A baby example

Consider maxwellian electromagnetism on an oriented riemannian 4-manifold X. A gauge field A is locally a 1 form and globally a U(1) connection. Its curvature F satisfies the Bianchi identity $dF = 0$ and the field equations in empty space assert $d * F = 0$. There is a dual gauge field A' whose curvature F' satisfies

$$F' = *F\,. \tag{1.1}$$

In the usual formulation we choose either A or A' as the fundamental field and write everything in terms of it. Electrically charged objects with respect to A are magnetically charged with respect to A' and *vice versa*.

As a toy model for self-dual fields, consider a formulation which includes both A and A' as fundamental fields. Thus the pair (A, A') functions as a self-dual field. (At the end of the introduction we remarked on field theories with self-dual fields.) Notice that the characteristic class of (A, A') is an element of the group $\Gamma(X) := H^2(X) \oplus H^2(X)$. To define the quantum theory[4] we need, as explained in [7], a symplectic form on $\Gamma(X)$ and a quadratic refinement of its reduction modulo two. Assume X is compact. The symplectic form is

$$\omega\left((x, x'), (y, y')\right) = x \cdot y' - y \cdot x' \tag{1.2}$$

and the quadratic refinement is

$$Q(x, x') = x \cdot x' \pmod 2\,. \tag{1.3}$$

In these expressions the dot is the cup product pairing followed by evaluation on the fundamental class. The two natural polarizations of $\Gamma(X)$ lead to the description of the partition function in terms of A or A'; the form Q vanishes in these cases.

E. Witten pointed out that Q is preserved by the $\mathrm{SL}(2;\mathbb{Z})$ action only if the intersection pairing is even, that is, only if X is spin. Thus the same is true of the partition function. This result is explained a different way in [8].

The standard euclidean kinetic term for a gauge field A is $\frac{1}{2}|F|^2$, which in local coordinates is $\frac{1}{4}\, F_{\mu\nu} F_{\mu'\nu'} g^{\mu\mu'} g^{\nu\nu'}$. (As usual g is the riemannian metric.) In the (A, A')

[4]We include these remarks, which we found instructive, even though we do not quantize self-dual gauge fields in our application to type-II superstring theory.

action there are kinetic terms for both A and A', so each appears in the action as 1/2 its usual value:

$$\int_X \left\{ \frac{1}{4}|F|^2 + \frac{1}{4}|F'|^2 \right\} \mathrm{vol}_X \,. \tag{1.4}$$

The extra factor of 1/2 is perhaps clearest in Minkowski spacetime, where the constraint (1.1) is imposed on the Euler-Lagrange equation.

2. Electric and magnetic charge

Continuing for the moment with the ordinary Maxwell theory of a single gauge field A, a particle has a worldline $W \subset X$, an oriented compact one-dimensional submanifold of X (not necessarily connected). Suppose the particle is electrically charged with respect to A. This is implemented by adding an interaction term to the euclidean lagrangian:

$$i \int_W q\, A\,. \tag{2.1}$$

Here q is a locally constant function on W which represents the number of units of charge carried by the particle. Charge quantization of the quantum theory asserts that q is integral (in appropriate units). Strictly speaking, only the exponential of (2.1) is well defined; it is interpreted as a power of the holonomy of the connection A. Topologically, q determines an element of $H^0(W)$. The contribution of the particle to the total electric charge on X is the pushforward of q in cohomology by the inclusion of W into X, which is a map $H^0(W) \to H^3_c(X)$. Here we take the image to be in *compactly supported* cohomology. In particular, the topological class of electric charge is an element of $H^3_c(X)$. At the level of differential forms, the electric charge is represented by a 3-form $j = j_{(W,q)}$ with compact support, the Noether current, and the field equation is modified to $d * F = j$. Strictly speaking, j is canonically defined only as a distribution supported on W, but as we will see it is often important to smooth it out. As explained in [1], the electric charge lies in the kernel of the natural map $H^3_c(X) \to H^3(X)$. Over $\mathbb{R}$ this follows from the field equation $d(*F) = j$.

Magnetic charge has a quite different classical description, and is most easily illustrated for a 0-form gauge field C in a three-dimensional spacetime Y (which we again take to be compact oriented riemannian). In other words, C is a circle-valued scalar field on Y. A particle may be magnetically charged with respect to C, so suppose $W \subset Y$ is a compact oriented one-dimensional submanifold, the worldline of a particle. As before, let q be a locally constant integer-valued function on W. Then to say the particle is magnetically charged with respect to C shifts the meaning of C. Usually one says that C is defined only on the complement of W and has winding number q around W. However, for many purposes it is not sufficient to have C defined on a subset of spacetime. In particular, it would not make sense then to restrict C

to W, something we need to do in the superstring theory context later on. Hence we need to explain in what geometric sense C extends over W. Obviously, C does not extend as a function over W. Instead, we represent the image of $[q]$ under the pushforward $H^0(W) \to H^2_c(Y)$ as the first Chern class of a circle bundle $P = P_{(W,q)}$ with smooth connection. Then C extends to all of Y not as a circle-valued function, but rather is a section of P. The field strength of C is now the "covariant derivative" of the section, more precisely the connection form relative to the section C. It is not closed; its differential is the curvature of P. Furthermore, as part of the construction of the "geometric magnetic current" P we may fix a trivialization on the complement of W. Then the ratio of C to that fixed trivialization, defined on the complement of W, is the usual description of a circle-valued gauge field with winding number q about W. Notice that although the Chern class of P is canonically determined from (W, q), the connection and curvature are not.[5] Returning to Maxwell theory in four dimensions, a magnetically charged particle worldline W with locally constant function q gives rise to a class in $H^3_c(X)$, which is now represented by a *gerbe* with connection, and the meaning of the gauge field A is now shifted in an analogous way by this gerbe. The connection on the gerbe is represented locally by a 2 form. (We discuss p-form fields in general in the next section.) As with electric charge, the topological class of magnetic charge is an element in the kernel of $H^3_c(X) \to H^3(X)$. The existence of A is a geometric form of the assertion that the image of magnetic charge in $H^3(X)$ vanishes. The curvature j of the gerbe is a 3-form, and at the level of differential forms the Bianchi identity is modified to $dF = j$.

If W is electrically charged with respect to A, then it is magnetically charged with respect to the dual gauge field A'. At the level of differential forms this follows from eq. (1.1) and the last equations in each of the two preceding paragraphs. A more precise argument including the geometric form of magnetic charge, at least in dimensions two and three, is given in [10, lecture 8].

Next, we simply observe that objects charged with respect to a self-dual gauge field carry electric and magnetic charge simultaneously. This is clear from the previous discussion for our toy model (A, A'): a particle electrically charged under A is magnetically charged under A', so both electrically and magnetically charged under (A, A'). An important subtlety here is the normalization of the coupling term. Since the kinetic terms (1.4) in the (A, A') action appear with $1/2$ the usual coefficient, so does the interaction term (2.1). The entire action is

$$\int_X \left\{ \frac{1}{4} |F|^2 + \frac{1}{4} |F'|^2 \right\} \mathrm{vol}_X + i \int_W \frac{1}{2} qA \,. \tag{2.2}$$

[5] We can, however, use the metric on X and a cutoff function to construct a *smooth* curvature form and also a connection with this curvature (as well as a trivialization on the complement of W). This leads to a coupling between the metric on X and magnetic charge which was the crucial idea in understanding the anomaly cancellation for the M-theory 5-brane [9].

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(Recall that in the classical theory we impose the equation $F' = *F$ in the (A, A') formulation.) One easy way to see the 1/2 in the coupling term is to compute the Euler-Lagrange equation from varying A — an overall factor in a classical lagrangian does not affect the equations of motion. We can also verify that in the quantum theory elimination of A' leads to the standard action for a charged particle. Namely, in (2.2) the particle is magnetically charged with respect to A', which is implemented by a shift in the geometric meaning of A'. Under duality that shift goes over into the electric coupling of A to the particle, and since the kinetic term for A' has 1/2 its usual coefficient, so does this coupling term under duality. Added to the coupling term already present in (2.2) we recover the standard action upon eliminating A', as claimed.

The factor of 1/2 in the coupling term is at first sight problematical, since for $q = 1$ the exponentiated action seemingly involves a square root of holonomy, which is anomalous. However, upon closer examination we find that in the self-dual formulation the periods of F, F' are *even*, and this extra factor of 2 renders the exponentiated action well defined. (One can see this factor of 2 in the quantization of the self-dual field using ϑ-functions, for example.)

As another example, consider a theory on a six-dimensional manifold X which includes a chiral 2-form gauge field B. If a 1-brane $W \subset X$, which is a closed oriented 2-dimensional submanifold, is electrically charged with respect to B, then it is also magnetically charged. Then we have two effects simultaneously: the meaning of B is shifted — in particular, H is not closed but rather its differential is a closed 4-form Poincaré dual to W — *and* there is a term

$$i \int_W \frac{1}{2} qB \tag{2.3}$$

included in the action.[6] Observe that (2.3) is of "Green-Schwarz type". In particular, it has an anomaly — its exponential is not a unit norm complex number, but rather is a unit norm element in an abstract complex line. Furthermore, if we study (2.3) as a function of parameters, we obtain a complex line bundle over the parameter space with metric and connection, and then (2.3) is a unit norm section which is not necessarily covariant constant. The line bundle is topologically trivial, but it may have non-zero curvature and/or holonomy. Such terms in actions contribute to the overall anomaly, which in geometric form is the curvature and holonomy of the tensor product of all such line bundles arising in the exponentiated action. Physicists usually express the anomaly by saying that terms like (2.3) are not invariant under gauge transformations (of the B-field).

[6] As in the self-dual (A, A') theory above, the quantization law for B involves a factor of 2 which makes (2.3) well defined. More precisely, the de Rham cohomology class of the field strength of B lies in the image of $H^3(X; \mathbb{Z})$ in $H^3(X; \mathbb{R})$ by *twice* the usual map.

3. Local fields and global topology

It is well known that fields which are locally represented by a differential p-form often have a global significance which is more intricate than a de Rham cohomology class. The first author initially encountered this in Chern-Simons theory [11];[7] in the physics literature it appears earlier in this context [12, 13] and implicitly even earlier in supergravity theories. The prototype is a 1-form gauge field which globally is a U(1) bundle with connection. Up to equivalence the p-form analog for higher p is a class in smooth Deligne cohomology [14, 15] or equivalently a Cheeger-Simons differential character [16]. But it is important in the physics that fields be defined on the nose, not simply up to equivalence. This is required by locality, for example, and is also necessary if one is to speak properly about group actions. Put differently, we need a framework which includes automorphisms of these fields. Furthermore, one also encounters "trivializations" or "sections" of this type of p-form field, as we did in the our discussion of magnetic charge.[8] A mathematical treatment including these refinements appears in ongoing work of the second author with I. Singer [17].[9] For our purposes here we simply remind that the set of equivalence classes of p-form fields analogous to U(1) connections on a manifold X, denoted[10] $\hat{H}^{p+1}(X)$, fits into the exact sequence

$$0 \longrightarrow H^p(X;\mathbb{R})/H^p(X;\mathbb{Z}) \longrightarrow \hat{H}^{p+1}(X) \longrightarrow A^{p+1}(X) \longrightarrow 0\,, \tag{3.1}$$

where

$$A^q(X) := \Big\{(\lambda,\omega) \in H^q(X;\mathbb{Z}) \times \Omega^q_{\text{closed}}(X) : \lambda_{\mathbb{R}} = [\omega]_{\text{de Rham}}\Big\}\,. \tag{3.2}$$

One should think of λ as the characteristic class and ω as the curvature, or field strength. The first term is the torus of topologically trivial flat elements. (A more detailed heuristic exposition of this type of p-form fields appears in [19, §6].)

There is a version of this story for any generalized cohomology theory. The generalized cohomology maps onto a full lattice in ordinary real cohomology. In physical applications of these ideas there are Dirac quantization conditions which dictate the proper lattice, and we choose a generalized cohomology theory accordingly.[11] The choice appropriate to RR fields in type II is K-theory. Then the basic exact sequence

[7] Embarrassingly, the two papers promised in [11] never appeared — there was trouble finding the proper context in which to work out the details of integration for global p-form fields of this type. The work of Hopkins and Singer cited below, as well as our forthcoming work mentioned in the introduction, fills this gap.

[8] Another important example in physics is the B-field in type-I supergravity coupled to super Yang-Mills — even in the classical theory without the Green-Schwarz term.

[9] For an outline of what is needed, see also [18, §6.3].

[10] We use the grading most natural in Deligne cohomology.

[11] The choice of map to real cohomology is also part of the Dirac quantization condition. For example, we noted above a factor of 2 in this map for certain self-dual fields.

for equivalence classes $\hat{K}^{p+1}(X)$ is

$$0 \longrightarrow K^p(X;\mathbb{R})/K^p(X) \longrightarrow \hat{K}^{p+1}(X) \longrightarrow B^{p+1}(X) \longrightarrow 0\,, \tag{3.3}$$

where

$$B^q(X) := \Big\{(x,\omega) \in K^q(X) \times \Omega^q_{\text{closed}}(X) : \mathrm{ch}(x) = [\omega]_{\text{de Rham}}\Big\}\,. \tag{3.4}$$

By Bott periodicity only the parity of p matters. There is a category $\mathcal{K}^{p+1}(X)$ whose equivalence classes are elements of $\hat{K}^{p+1}(X)$.

The Ramond-Ramond fields of type-II superstring theory (with vanishing B-field), taken together in all degrees, are objects in $\mathcal{K}^q(X)$. (The parity of q is even for type IIA and odd for type IIB. The self-duality condition on the field strength is imposed when quantizing these fields, which we do not do in this paper.) This is the precise version of the proposal made in [1], particularly their equation (2.17), which asserts that (up to a factor of 2π) the field strength G of an RR field which maps to $(x,\omega) \in B^\bullet(X)$ is[12]

$$G = \sqrt{\hat{A}(X)}\,\omega\,. \tag{3.5}$$

Several motivations for this proposal were explained there. Another idea which originally motivated that equation is the following: When one carries out the constructions of field theory with these new local objects — objects in $\mathcal{K}^\bullet(X)$ — the geometric meaning of certain quantities changes. In particular, the Noether currents associated to gauge symmetries will now also be elements of $\mathcal{K}^\bullet(X)$, but with opposite parity. This makes it natural that the Noether charge, the topological equivalence class of the Noether current, is an element of the appropriate K-theory group, as in [20].

As mentioned, there is a geometric version of any generalized cohomology theory. We will make use of this for KO-theory (real bundles) and KSp-theory (quaternionic bundles). We term such theories *differential* K-theory, *differential* KO-*theory*, etc.

4. Pfaffians of Dirac operators

This is well-known material.

The functional integral over a free spinor field is formally the pfaffian of a Dirac operator, which is an element of a real or complex line. If $W \to S$ is a family of

[12] Here we interpret $\sqrt{\hat{A}(X)}$ as a differential form. Note that it is invertible (of the form 1 + nilpotent). One explanation of its appearance is the following. On a compact spin^c manifold X, both $K^\bullet(X)\otimes\mathbb{R}$ and $H^\bullet(X;\mathbb{R})$ carry an addition, multiplication, and a bilinear form. The Chern character preserves the addition and multiplication but not the bilinear forms. The modification by $\sqrt{\hat{A}(X)}$, as in (3.5), preserves addition and the bilinear forms but not the multiplication. The physics uses the addition (superposition of states in quantum mechanics) and the bilinear form (for example, in the coupling term (5.1) below), but not as far as we can tell the multiplication.

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n-dimensional spin manifolds, then the pfaffian is a section of a real or complex line bundle with metric and connection over S. We consider Dirac operators with coefficients in a bundle $E \to W$, which may be real, complex, or quaternionic. Now up to equivalence a real line bundle with metric and connection over S is an element of $H^1(S;\mathbb{Z}/2\mathbb{Z})$; in the complex case it is an element of $\hat{H}^2(S)$. The construction of these bundles depends on $n \pmod 8$, and we quickly review the constructions in the language of the previous section. (For a description of the even-dimensional case, see [21]; for the odd-dimensional case, see [22] and also [1, §4.1].)

A geometric form of the index theorem asserts that the index of the family of Dirac operators is the image of E under integration (pushforward) in differential K-theory, which is a map

$$\hat{K}(W) \longrightarrow \hat{K}^{-n}(S)\,. \tag{4.1}$$

We then compose with a determinant or pfaffian to obtain the appropriate line bundle. The map (4.1) has refinements in real and symplectic K-theory, and we need these to obtain pfaffians.

The even-dimensional case goes as follows. For $n \equiv 0, 4 \pmod 8$ we may as well suppose the bundle E is complex, in which case by periodicity the image of E under (4.1) lies in $\hat{K}(S)$. The *determinant* line bundle is obtained by a map $\hat{K}(S) \to \hat{H}^2(S)$ which refines the usual topological determinant line bundle in K-theory. In quantum field theories CPT invariance dictates that if there are positive half-spinors with values in E, then there are also negative half-spinors with values in $\overline{E}$. Formally the fermionic path integral is the *pfaffian* of $D(E-\overline{E})$, Dirac coupled to the formal difference $E-\overline{E}$. But the determinant line bundle $\operatorname{Det} D(E-\overline{E})$ is isomorphic to $\operatorname{Det} D(E)^{\otimes 2}$, so we take $\operatorname{Pfaff} D(E-\overline{E})$ to be $\operatorname{Det} D(E)$. For $n \equiv 2 \pmod 8$ we obtain a pfaffian if E is real. Then the index lies in $\widehat{\mathrm{KO}}^{-2}(S)$, and there is a pfaffian line bundle $\widehat{\mathrm{KO}}^{-2}(S) \to \hat{H}^2(S)$. Similarly, for $n \equiv 6 \pmod 8$ if we start with a quaternionic bundle E, then we apply (4.1) to $\widehat{\mathrm{KSp}}$ thereby obtaining an index in $\widehat{\mathrm{KSp}}^{-6}(S) \cong \widehat{\mathrm{KO}}^{-2}(S)$ and so a pfaffian as before.

In the odd-dimensional case the complex Dirac operator is self-adjoint, but the determinant is naturally a complex number — the exponentiated η-invariant enters. The square root is then a section of a real line bundle, and the construction of this bundle is topological. For $n \equiv 1 \pmod 8$ we start with a real bundle E. The index lies in $\mathrm{KO}^{-1}(S)$, and the pfaffian line bundle is obtained by the natural map $\mathrm{KO}^{-1}(S) \to H^1(S;\mathbb{Z}/2\mathbb{Z})$. For $n \equiv 3 \pmod 8$ we also start with a real bundle, and again there is a natural map $\mathrm{KO}^{-3}(S) \to H^1(S;\mathbb{Z}/2\mathbb{Z})$ (which factors through a map $\mathrm{KO}^{-3}(S) \to H^1(S;\mathbb{Z})$). For $n \equiv 5, 7 \pmod 8$ we take E to be quaternionic, and by periodicity we obtain the same constructions of the pfaffian.

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5. Coupling to D-branes

The spacetime X of type-II theory is a spin riemannian 10 manifold, which for simplicity we assume to be compact. The worldvolume W of a D-brane is a submanifold endowed with a complex vector bundle $Q \to W$. There are also scalar fields and fermions on the brane. In the basic case this Chan-Paton bundle Q has rank one, corresponding to unit charge, and as demonstrated in [19] is more properly viewed as a spin^c structure on W. For simplicity we assume that W is spin, and allow for arbitrary rank vector bundles Q. In fact, Q comes equipped with differential geometric data and should be viewed as an element of $\mathcal{K}^0(W)$.

Finally we are ready to describe the coupling of the D-brane to the RR-field $C \in \mathcal{K}^q(X)$. Since the field strength of C is self-dual, the D-brane is both electrically and magnetically charged under C. The magnetic charge means that the geometric meaning of C is shifted. Namely, the D-brane charge is represented by the pushforward of Q to the bulk X, which is an element of $\mathcal{K}^p(X)$, where p is the codimension of W in X. As explained earlier, the precise construction of a *smooth* element depends on some choices. Then C is a trivialization of this element which obeys the following constraint. The restriction of the D-brane charge to W is Q times the Euler class of the normal bundle to W in K-theory. If $p \equiv -1, 0, 1 \pmod 8$, then this Euler class has a real refinement, and we constrain $\overline{Q}\, C$ to be a trivialization of $\overline{Q}\, Q$ times this real refinement. If $p \equiv 3, 4, 5 \pmod 8$, then this Euler class has a quaternionic structure, and we constrain $\overline{Q}\, C$ to be compatible.[13] As we explained after eq. (2.2) there is an extra factor of $1/2$ in the electric and magnetic coupling of a self-dual field. The constraint just described is our interpretation of this factor in the magnetic charge. The electric coupling is a term in the action:

$$i \int_W \frac{1}{2}\, \overline{Q}\, C\,. \tag{5.2}$$

The interpretation of this term depends on the dimension. For $p \equiv -1, 0, 1 \pmod 8$ we interpret the exponential of (5.2) as $\exp(i \int_W \overline{Q}\, C)$ in $\widehat{\mathrm{KO}}$-theory. The cases

[13] These statements requires a technical explanation, details of which will appear in our subsequent paper. Let $\nu \to W$ be the normal bundle, which is real of rank p. If p is even, then the K-theory Euler class in $K(W)$ may be identified with the difference of the half-spinor bundles of ν. It has a real structure if $p \equiv 0 \pmod 8$, a quaternionic structure if $p \equiv 4 \pmod 8$, and is complex (not self-conjugate) if $p \equiv 2, 6 \pmod 8$. In these cases we simply regard the complex K-theory Euler class as an element of the appropriate differential KO-, KSp-, or K-group. If p is odd, the Euler class in topological K-theory vanishes, but in differential K-theory it is an element of order 2 in $K^0(X;\mathbb{R})/K^0(X) \hookrightarrow \hat{K}^1(X)$. It can be identified with the mod2 reduction of the spinor bundle $S(\nu)$. We summarize the Euler class computation in complex differential K-theory by the equation

$$\mathrm{Euler}_{\hat{K}}(\nu) = \begin{cases} S^+(\nu) - S^-(\nu)\,, & \mathrm{rank}(\nu) \text{ even}\,; \\ S(\nu) \pmod 2\,, & \mathrm{rank}(\nu) \text{ odd}\,. \end{cases} \tag{5.1}$$

$p \equiv 3,4,5 \pmod 8$ are similar, except that we use $\widehat{\mathrm{KSp}}$-theory. For $p \equiv 2,6 \pmod 8$ the square of the exponential of (5.2), computed in $\hat{\mathrm{K}}$-theory, is a section of a line bundle which is a square, as explained in the previous section. We take (5.2) to be one of the square roots of that section.

As in (2.3) this term is of Green-Schwarz type — its exponential is a section of a line bundle with connection over the space of parameters. Thus it contributes to the overall anomaly. Let $W \to S$ be a family of D-branes parametrized by a manifold S. This comes as a fiberwise submanifold of a family $X \to S$ of 10-dimensional spacetimes, which we do not require be a product. The line bundle in question is computed by integrating $\overline{Q}\,Q \cdot \mathrm{Spinors}(\nu)$ over the fibers of $W \to S$ in the appropriate differential K-theory. As explained in the previous paragraph, we identify the differential K-theory Euler class with "Spinors(ν)", which if p is odd is identified with the spin bundle of ν modulo two and if p is even is the difference of half-spin bundles. In all cases it is to be regarded in the appropriate $\hat{K}$-, $\widehat{\mathrm{KO}}$-, or $\widehat{\mathrm{KSp}}$-group. (For $p \equiv 2,6 \pmod 8$ we use the square root mentioned at the end of the previous paragraph.)

The (complex) Fermi field on W is the restriction of a chiral spinor in X to W tensored with $\mathrm{End}(Q) \cong \overline{Q}\,Q$. Since we assume W is spin, the restriction to W of the bundle of half-spinors on X decomposes. If p is odd we obtain the tensor product of spin bundles $S(W) \otimes S(\nu)$, whereas if p is even we have the sum $S^+(W) \otimes S^+(\nu) \oplus S^-(W) \otimes S^-(\nu)$. The fermionic functional integral — Dirac pfaffian — is computed by coupling Dirac to $S(\nu)$ in the odd case and the formal difference $S^+(\nu) - S^-(\nu)$ in the even case. As explained in the previous section, the pfaffian line bundle is computed by the Atiyah-Singer index theorem: we regard these spin bundles in the appropriate differential complex, real, or quaternionic K-group and integrate. Therefore, we get precisely the same line bundle with connection as in the previous paragraph. Presuming that the coupling term (5.2) comes with an overall minus sign, we see that the anomaly from the fermions on the brane cancels the anomaly from (5.2).

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